The Development of Mathematics in Medieval Europe by Menso Folkerts

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This is Menso Folkerts' second Variorum volume. The first was published in 2003 [see Høyrup 2007b for a review]; it contained papers dealing with the properly Latin tradition in European mathematics, that is, the kind of mathematics which developed (mainly on the basis of *agrimensor* mathematics and the surviving fragments of Boethius' translation of the *Elements*) before the 12th-century Arabo-Latin and Greco-Latin translations. This second volume deals with aspects of the development which took place after this decisive divide, from *ca* 1100 to *ca* 1500.

Few scholars, if any, know more than Folkerts about medieval Latin mathematical manuscripts. It is, therefore, natural that the perspective on mathematics applied in the papers of this volume is on *mathematics as a body of knowledge*, in particular, as it is transmitted in and between manuscripts. To the extent that *mathematics as an activity* is an independent topic, it mostly remains peripheral, being dealt with through references to the existing literature—exceptions are the investigations of what Regiomontanus and Pacioli *do* with their Euclid [in articles VII and XI]—or it is undocumented, as when it is said that Jordanus de Nemore's *De numeris datis* was 'probably used as a university textbook for algebra' [VIII.413]. There should be no need to argue, however, that familiarity with the body of mathematical knowledge is fundamental for the study of mathematics from any perspective: whoever is interested in medieval Latin mathematics can therefore learn from this book.

It is more questionable that Folkerts tends to describe the mathematics which he refers to through their modern interpretation. To say, for instance, that the *Liber augmentis et diminutionis* shows

'how linear equations with one unknown or systems of linear equations with two unknowns may be solved with the help of the rule of double false position' [I.5] does not help the reader who is not already familiar with the kind of problems to which this rule was applied to understand that the treatise contains no equations but problems which modern scholars often explain in terms of linear equations.[1]

Since many of the articles are surveys, they touch by necessity on topics outside Folkerts' own research interest. In such cases, Folkerts tends to mention existing disagreements or hypotheses instead of arguing for a decision (even in cases where one may suspect that he has an opinion of his own).[2] This is certainly a wise strategy, given the restricted space for each topic; but the reader should be aware that this caution does not imply that existing sources do not allow elaboration or decision.

With one exception, all articles in the volume turn around the tradition and impact of the *Elements*, and/or the figure of Regiomontanus. Unlike many Variorum volumes, several articles are not pub-

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[1] For instance,

> Somebody traded with a quantity of money, and this quantity was doubled for him. From this he gave away two dragmas, and traded with the rest, and it was doubled for him. From this he gave away four dragmas, after which he traded with the rest, and it was doubled. But from this he gave away six dragmas, and nothing remained for him. [Libri 1838–1841, 1.326]

Seeing this simply as 'an equation' also misses the point that it may just as legitimately be seen (for example) as a system of three equations with three unknowns (the successive amounts traded with).

Actually, the treatise solves this problem (and many others) not only through application of a double false position but also by reverse calculation and by means of its *regula* (which Fibonacci calls the *regula recta*, first-degree *res*-algebra).

[2] In I.n13, it is said that the author of a reworking of al-Khwārizmī's algebra could be Guglielmo de Lunis. This hypothesis is quite widespread. It is not mentioned that the only two independent sources which inform us about a translation of this work (whether Latin or Italian) made by Guglielmo (Benedetto da Firenze and Raffaello Canacci, Lionardo Ghaligai depending on Benedetto), both quote it in a way which appears to exclude the identification of Guglielmo de Lunis as its author. I guess Folkerts knows both sources.

lished in their original form but have been rewritten so as to encompass recent results. In total, 12 articles are included.

I. 'Arabic Mathematics in the West'

This revised translation of a paper originally published in German in 1993 deals with the arithmetic of Hindu numerals, algebra, Euclidean geometry (*Elements*, *Data*, *Division of Figures*), spherics, and other geometrical topics (Archimedean works on the circle and the sphere, conics, practical mensuration). Given its brevity (16 pp.), this is obviously little more than a (very useful) bibliographic survey.

II. 'Early Texts on Hindu-Arabic Calculation'

This article (26 pp.), which was first published in 2001, falls into two parts. The first part (6 pp.) is a general survey covering the Indian introduction of the decimal place value system and its diffusion into the Arabic world, some of the major Arabic texts describing the system, the early Latin redactions of *Dixit algorizmi*, and the most important Latin algorism texts from the 13th and 14th centuries. The second part (17 pp.) is a detailed description of *Dixit algorizmi*, the earliest Latin reworking of the translation of al-Khwārizmī's treatise on the topic. Of this reworking, two manuscripts exist; the second one was discovered by Folkerts, who also published a critical edition [Folkerts 1997].

III. 'Euclid in Medieval Europe'

This is a completely revised version (64 pp.) of a paper first published in 1989. The first half of the article describes all known medieval European translations and redactions from Boethius until the mid-16th century; it also includes a brief discussion of the Arabic versions. The second half is a 'list of all known Latin and vernacular manuscripts up to the beginning of the 16th century that contain the text of Euclid's *Elements* or reworkings, commentaries, and related material'.

IV. 'Probleme der Euklidinterpretation und ihre Bedeutung für die Entwicklung der Mathematik'

This article (32 pp.) was originally published in 1980. An initial section covers the same ground as the first part of article III, but with more emphasis on the character of the various versions of the *Elements*. Sections 2 and 3 look at how late ancient as well as Arabic and Latin commentators and mathematicians concentrated on specific aspects of the *Elements*: proportion theory, the parallel postulate, the theory of irrationals.

V. 'Die mathematischen Studien Regiomontans in seiner Wiener Zeit'

This paper (36 pp.) was originally published in 1980. It deals with a phase in Regiomontanus' mathematical development of which little had been known. In Folkerts' words, it shows that

> laborious work on details may still allow one to find many mosaic cubes which, admittedly, do not change the picture of Regiomontanus the mathematician completely, but still allows making it much more distinct. [V.175–176]

At first, Folkerts analyzes Regiomontanus' *Wiener Rechenbuch*, a manuscript from Regiomontanus' hand written between 1454 and *ca* 1462 (Codex Wien 5203), containing original work as well as borrowed texts (at times, however, apparently rewritten in Regiomontanus' own words). Next, Folkerts traces which treatises on *Visierkunst* (the practical mensuration of wine casks) Regiomontanus must have possessed or known, using the posthumous catalogues of Regiomontanus' library and those parts of the codex Plimpton 188 which once belonged to Regiomontanus. Finally, Folkerts digs out from the same Plimpton codex evidence that the algebraic knowledge which Regiomontanus displays in his correspondence with Bianchini and others was already his in 1456 (including matters which are now known to have been current in Italian 14th-century *abbaco* algebra but not found in the *Liber abbaci* nor in al-Khwārizmī). Even the symbolism that Regiomontanus uses after 1462 turns up in the Plimpton codex, both in passages that stem from Regiomontanus' hand and in others for which he is probably not responsible.

VI. 'Regiomontanus' Role in the Transmission and Transformation of Greek Mathematics'

This article (26 pp.) was originally published in 1996. After some biographical information, it presents Regiomontanus' 'programme', that is, the leaflet listing the works which Regiomontanus intended to print on his own press (plans that were never realized because of his sudden death). Beyond some of Regiomontanus' own writings, it includes in particular the *Elements*, Archimedes' works, Menelaus' and Theodosius' spherics, Apollonius' *Conics*, Jordanus de Nemore's *Elements of Arithmetic* and *On Given Numbers*, Jean de Murs' *Quadripartitum numerorum* and his *Algorismus demonstratus*. The 'programme' is supplemented by Regiomontanus' Padua lecture from 1464, which refers to many of the same works and also to Diophantus. Next, Folkerts uses manuscripts which were demonstrably in Regiomontanus' possession, his annotations, and so forth, to determine how much Regiomontanus actually knew about the authors and works he mentions—which was indeed much. Only in the case of the *Conics* is it not certain that he was familiar with more than the beginning of the work as translated by Gherardo da Cremona.

The final pages of this article present various numeric, geometric, as well as determinate and indeterminate algebraic problems not coming from Greek sources but present in: the *Wiener Rechenbuch*, a problem collection in the Plimpton manuscript (in Regiomontanus' hand and apparently from 1456), the manuscript *De triangulis*, and the letters exchanged with Giovanni Bianchini, Jacob von Speyer, and Christian Roder. Some of the geometric problem solutions make use of algebraic techniques.

The discussion of approximations to the square root of a number $n = a^2 + r$ on VI.109 invites comment. The *Rechenbuch* as well as the Plimpton collection offer the usual first approximation $\sqrt{n} \approx n_1 = a + r/2a$. The Plimpton collection then gives a second, supposedly better, approximation

$$n_2 = a + \frac{4a^2 + 2r - 1}{(4a^2 + 2r) \cdot 2a}$$

about which Folkerts says that it is not clear where it comes from. Actually, the formula is wrong—it reduces to

$$a + \frac{4a^2 - 1}{8a^3}$$

when $r = 0$, not to a. However, iteration of the procedure which yields n_1 gives

$$\tilde{n}_2 = a + \frac{(4a^2 + 2r) \cdot r - r^2}{(4a^2 + 2r) \cdot 2a}$$

which coincides with the Plimpton second approximation for $r = 1$. In the present context, one might have expected that Regiomontanus dealt only with an example where $r = 1$, and that the general formula as such is a reconstruction due to Folkerts. However, in VIII.422, Regiomontanus is quoted for the observation that the second approximation cannot be applied to all numbers, which is obviously not true for the approximation $\tilde{n}_2$. Regiomontanus must, therefore, be presumed to be at least co-responsible for the mistake.

Folkerts quotes the *Rechenbuch* for a different second approximation, *viz*

$$n_2 = \frac{n}{n_1} : 2.$$

This is obviously a misprint for

$$n_2 = \left(n_1 + \frac{n}{n_1} \right) : 2.$$

By the way, a bit of calculation shows that this n_2 and what was called $\tilde{n}_2$ above are algebraically equivalent.

VII. 'Regiomontanus' Approach to Euclid'

This paper (16 pp.) is a completely revised translation of an article first published in German in 1974. Its first half elaborates in greater depth the Euclidean aspect of the previous article and the presentation of the posthumous catalogues of Regiomontanus' *Nachlaß* from article V. The second half analyses Regiomontanus' endeavor 'to establish a correct text of Euclid' which was mainly based on mathematical critique of the Campanus version but also drew on 'Version II' (formerly known as 'Adelard II'). As summed up by Folkerts [VII.10], Regiomontanus' aim was 'to establish a mathematically correct text

(not to be understood in modern text-critical sense of a reconstruction of the original text)', as was indeed 'typical for Regiomontanus'.

VIII. 'Regiomontanus' Role in the Transmission of Mathematical Problems'

This article (18 pp.) was first published in 2002. It broadens the range of problem types with respect to those discussed in the end of article VII, and says more about the way in which the problems are solved. The sources are the Plimpton problem collection, the correspondences, and the *Wiener Rechenbuch*. In particular, a number of problems going back to the Italian *abbaco* tradition are presented.

Several of these problems turn up again in the following decades in mathematical writings from southern Germany, first in a manuscript copied by Fridericus Amann in 1461—at times with the same numerical parameters. Folkerts concludes that 'Fridericus Amann must have learned something of the contents of MS Plimpton 188 soon after it was finished' [VIII.414], and that 'Regiomontanus played a crucial role in transmitting mathematical knowledge from Italy to Central Europe in the 15th century.' Given that even the problems in the Plimpton manuscripts are copied from an earlier source, this seems to me to be a daring conclusion to say the least.[3]

Some observations should be made. First, on VIII.418 it is stated that nos. 16–32 of the Plimpton collection ask for a number and serve as examples for al-Khwārizmī's six problem types. This seems to be a typographical mistake (for 16–21?).[4] Next, the erroneous second-order approximation to a square root from the Plimpton collection is repeated on VIII.422, whereas the one from the *Rechenbuch* is correct this time. Finally, on VIII.419, something is wrong in the presentation of a 'special arithmetical problem'— probably already in the original.[5]

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[3] See 136n17 below, and preceding text.

[4] According to Folkerts, no. 22 deals with compound interest (but illustrates al-Khwārizmī's fourth type), and nos. 27 and 30 are, respectively, of the types 'purchase of a horse' and 'give and take.'

[5] The problem from the Plimpton collection states that 'somebody wants to go as many miles as he has dinars. After every mile the dinars he possesses are doubled, but he loses 4 dinars. At the end he has 10 dinars.' Folkerts solves

IX. 'Leonardo Fibonacci's Knowledge of Euclid's *Elements* and of Other Mathematical Texts'

This article (25 pp.) was still to appear when the present volume was prepared (it was eventually published in the Fall of 2005). Going through the *Liber abbaci*, the *Pratica geometrie*, the *Flos*, the letter to Master Theodorus, and the *Liber quadratorum*, Folkerts traces the mathematical works that are used with 'due reference' as well as those which are used without recognition of the borrowing. Euclid is quoted very often; Archimedes, Ptolemy, Menelaus, Theodosius, and the *agrimensores*, occasionally; but Arabic authors are not cited at all (with the sole exception of *Ametus filius*, i.e., Amad Ibn Yūsuf).[6]

The last part of the article raises the question 'Which version of Euclid did Leonardo use?' Often Fibonacci seems to quote from memory—the same proposition may be formulated in different words in the *Liber abbaci* and the *Pratica*, none of the formulations agreeing with any known Latin or Arabic version. Elsewhere, it is clear that Fibonacci uses the Latin translation from the Greek.

X. 'Piero della Francesca and Euclid'

This article (22 pp.) was first published in 1996. It starts by sketching the story of the Arabo-Latin *Elements* (with emphasis on Campanus) and by giving a brief general description of Piero's mathematical

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this without making use of the magnitude of the remainder (the algebra involved cannot correspond to anything Regiomontanus would do), finding that the man starts with 4 dinars—but in that case he will be left with 4 dinars after each doubling and subtraction, never with 10. Regiomontanus has a marginal note that the problem has to be solved 'in a reversed order', which Folkerts suggests might mean by 'trial and error'. This is not likely: stepwise backward calculation was a standard method for such 'nested-box' problems. Going backwards from 10 dinars, we get the successive remainders $7, 5\frac{1}{2}, 4\frac{3}{4}, 4\frac{3}{8}, \ldots$. The data of the problem are thus inconsistent (if rendered correctly), which Regiomontanus does not seem to have noticed.

[6] Since Fibonacci asserts regularly that his methods are of Arabic origin, this *could* mean that he made his apparent borrowings from Abū Kāmil, al-Karajī, and others indirectly. However, his obvious verbatim copying from Gherardo da Cremona's translations of al-Khwārizmī [Miura 1981] and Abū Bakr [Høyrup 1996, 55] weakens the argument—at times, Fibonacci clearly did not want to reveal his sources.

works based on Davis 1977. Turning then to the use of Euclid, Folkerts shows that even Piero is fond of citing Euclid (mostly the *Elements*, but in *De prospective pingendi* the *Optics* as well). There is
no doubt that Piero used the Campanus version—he cites Campanus
twice and uses some of his additional propositions. However, Piero's
words and terminology often differ from those of Campanus in a way
which reflects Piero's background in the *abbaco* tradition—both in
the *Libellus de quinque corporibus regularibus*, which was originally
written in Italian but is only extant in Latin translation, and in the
Trattato d'abaco. Folkerts supposes this to reflect lack of familiarity
'with the style used in scientific mathematical works' [X.302] and not
the use of a non-Campanus version. He points out that Piero's numbering of certain propositions from book 15 show that the manuscript
he used is not among those known today.

Article X concludes by examining the citations of Vitruvius,
Ptolemy, Archimedes, and Theodosius in Piero's mathematical writings as well as the possible sources for his treatment of semiregular
solids—for which Jean de Murs' *De arte mensurandi* might be one
but not the only source.

XI. 'Luca Pacioli and Euclid'

This article (13 pp.) was originally published in 1998. Within the
framework of a short biography concentrating on Pacioli's interaction with Euclid, it discusses the traces of his translation of Euclid
into the vernacular, the excerpts from the *Elements* in the *Summa
de arithmetica geometria proportioni et proportionalita* from 1494
(drawn from the Campanus tradition), and his Latin edition of a
purportedly restored Campanus text in 1509.

The vernacular translation turns out to have probably been
made before the first part of the *Divina proportione*, i.e., before 1497.
The arithmetical part of the *Summa* contains excerpts from *Elements*
5;[7] the geometrical part excerpts from books 1–3, 6 and 11. The material is transformed in a way which was presumably suited for a public
with practical but only modest theoretical interests: the Euclidean
material is brought in the beginning of sections—thus serving as 'the-

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[7] These excerpts, dealt with previously by Margherita Bartolozzi and Raffaella
Franci [1990], are not discussed further by Folkerts.

oretical' underpinning for what follows—but there is no clear separation between definitions and enunciations, and proofs are mostly replaced by explanations with reference to diagrams.

The definitions from book 1, as well as all excerpts from book 11, are rendered rather freely. The rest of the excerpts from book 1 as well as those from books 2–3 and 6 are very close to the Campanus text. They cannot have been taken over from Pacioli's vernacular translation, since they agree rather precisely with passages in the manuscript BN Florence, Palatino 577, probably from *ca* 1460.[8]

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[8] This agreement appears from the presentation to have been established/ checked by Folkerts himself. For the statement that the 'geometrical section of Pacioli's *Summa* agrees in the other parts, too, with that Florence manuscript' [XI.226], Folkerts refers to Picutti 1989.

Because of the widespread, unconditional acceptance of the thesis of this paper, which is meant to convince readers that Pacioli, *in claimed contrast to other abacus writers*, was a vile plagiarist, the reviewer would like to make some observations. Picutti's paper is written in a strong and explicitly anticlerical key, which may be quite understandable in an Italian context, but is in itself no argument for its reliability—nor of course for the opposite. (Compare Libri's wonderfully and similarly engaged *Histoire des sciences mathématiques en Italie* [1838–1841], which is still valuable after more than 150 years). So, without further evidence, one should probably not follow an author who claims that Pacioli divides his text into chapters instead of 'distinctions' [Picutti 1989, 76]. Actually, the chapters are subdivisions of the distinctions, the distinctions are indicated in the titles, and the actual distinction as well as the chapter are indicated in the running head of all pages, in the 1494 edition of the *Summa* as well as the second edition from 1523. Picutti seems not to have examined any of them seriously. (Without endorsing peer-review hysteria, the reviewer also asks himself why Picutti only published in the Italian edition of *Scientific American* and never substantiated his assertions in a professional journal.)

On the other hand, it is obvious from a reproduced passage that Pacioli sometimes used either Palatino 577 or a precursor manuscript. Since Pacioli has diagrams which are omitted in the Palatino manuscript (as admitted by Picutti), it is plain that Pacioli either used this manuscript creatively or that he borrowed from a precursor where the diagrams were present (the one shown in the reproduction is not in Fibonacci's *Pratica*, at least not in Boncompagni's edition [1862]). Elsewhere in the *Summa*, however, misprints in the lettering of the diagrams can be corrected by means of the Boncompagni edition of the *Pratica*. Pacioli evidently felt free to copy without acknowledging his sources explicitly, while stating in the initial unfoliated *Sommario*

Folkerts' comparison of Pacioli's edition of the Campanus text with the *editio princeps* from 1482 shows that the proper corrections are minor, and that the main difference consists in the addition of comments introduced by the word *castigator* (which suggests that they were meant to be understood as corrections). In total, Folkerts counts 136 additions, 42 of which are more than 10 lines long. For the most part, 'Pacioli confines himself to explaining terms or individual steps within a proof or construction' [XI.228]; at times, he 'makes remarks that are not immediately necessary for the understanding of the theorem, but are suggested by it' [XI.229]. So, we may assume 'that the edition of Euclid contained elements from Pacioli's mathematical lectures' [XI.230].

XII. 'Algebra in Germany in the Fifteenth Century'

This article (18 pp.) has not been published before. Its theme was already touched on in articles V, VI, and VIII; but here the perspective is broadened. Some of the essential sources for the arguments have been published but much material remains in unpublished manuscripts, and a survey like the present one is certainly needed, if only to create a context for further research.

The article starts by presenting the background in Italian *abbaco* algebra. This account, as explained, is built on Franci and Toti Rigatelli 1985, which must now be considered partially outdated.[9] The claim [XII.3] that Piero della Francesca 'contributed not only to perspective but also to algebra', and that therefore and for other reasons Luca Pacioli 'has enjoyed unmerited fame, for his algebra contains nothing new of any value' is unwarranted. After all, Piero—

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that most of his volume has been taken from Euclid, Boethius, Fibonacci, Jordanus, Blasius of Parma, Sacrobosco, and Prosdocimo de' Beldomandi. Fibonacci, Piero, and many other writers in the *abbaco* traditions borrowed as freely and gave neither specific nor general reference when the name of the source carried no prestige. Only renewed scrutiny of the Palatino manuscript will reveal whether Pacioli also copied directly from Fibonacci's *Pratica* or *only* indirectly.

[9] Its aim was 'to shed light on the algebraic achievements of the Italian algebraists of the Middle Ages, rather than to investigate their sources and internal links' [Franci 2002, 82n2]; it even precedes a paper [Franci and Toti Rigatelli 1988] which the authors characterize as a 'first summary'.

truly impressive as he is as a geometer—repeated without distinction traditional nonsense along with valuable material in his algebra: he obviously copied texts without checking or making calculations. Pacioli reflected on the algebraic material that he borrowed, exactly as he reflected on his Euclidean borrowings.[10]

The treatment of Germany begins with a presentation of Regiomontanus' contributions, with particular emphasis on his symbolism. Its thesis is that Regiomontanus 'was central for the transmission of Italian ideas about algebra to Central Europe' [XII.3].

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[10] Piero repeats those false rules for higher-degree equations which had circulated at least since Paolo Gherardi (1328). See, for instance Arrighi 1970, 13 on solving the problem 'cubes equal to things and number' (in modern symbols, $\alpha x^3 = \beta x + n$) as if it had been '*censi* equal to things and number' ($\alpha x^2 = \beta x + n$). Rules which hold in specific cases only (as pointed out by Dardi da Pisa in 1344) are stated by Piero as universally valid—see, for example, Arrighi 1970 146. Piero also copies a long sequence of rules for quotients between algebraic powers, in which 'roots' take the place of negative powers, the first negative power being identified with 'number' (the rules appear to go back to a treatise written by Giovanni di Davizzo in 1339) [cf. Høyrup 2007c and Giusti 1993, 205]. See also Enrico Giusti's characterization of the algebraic Piero as

> a copyist who does not even notice—witness the very high number of repetitions of cases that were already treated (13 out of a total of 61)—that what he was writing had already been copied one or two pages before,

and as 'an author... who did little more than to collect whatever cases he might find in the various authors at his disposition, without submitting them to accurate examination' [Giusti 1991, 64 (trans. JH)].

Pacioli points out explicitly [1494, 1.150r] that no generally valid rule had so far been found for cases where the three algebraic powers are not separated by 'equal intervals'. (He was not the first to point it out: a similar observation is made in the *Latin algebra* [Wappler 1887, 11]—see 135n16 below and pertinent text). Pacioli also stays aloof of the confusion between negative powers and roots. He does include [1494, 1.67v, 143r–v] a terminology where 'nth root' stands for the $(n-1)$th (positive) power of the *cosa*. But, since this system identifies the 'first root' with the *cosa*, it is likely to be an outgrowth of the al-Khwārizmīan use of root (namely the square root of the *māl/census*) for the first power—an outgrowth of which Pacioli is *not* the inventor, since he describes the system for completeness' sake.

According to Folkerts, Regiomontanus uses the following symbols or abbreviations:[11]

○ a superscript r or R provided with a curl to indicate an abbreviation for *res* or *radix*, following after the coefficient,
○ a superscript c also provided with a curl and following the coefficient, for '*census*',
○ a long horizontal stroke connecting the two sides of the equation (which may thus be read as an equality sign in the function of equation sign),
○ a sign for minus that has been interpreted as $\bar{\imath}$ (that is, *in*) followed by the curl meaning *us*, ɩɋ̄.[12]

However, the shapes shown in a photo in Cajori 1928–1929, 1.96 from the calculations made for a letter to Bianchini—*viz* 𝔪̂, at times becoming 𝔪̂—look more like pen variants of the traditional Italian shape 𝔪̃,[13] while a page from the Plimpton manuscript[14] uses the shape 𝔪̂ twice but the shape ⟿� (meaning *mī(us)*) four times. The same page shows the abbreviation for *res* superscripted once but more often on the line (and even more often with the full word *cosa*). All in all, Regiomontanus symbols (mostly used as mere abbreviations) are much less fixed than Folkerts' description would have us believe.

In his Vienna period, as pointed out, Regiomontanus copied al-Khwārizmī's algebra (in Gherardo's translation) and Jean de Murs' *Quadripartitum numerorum*, and annotated both carefully. As concerns the algebraic problems contained in the Plimpton collection, *De triangulis*, and the correspondences, Folkerts restricts himself *grosso modo* to a cross-reference to articles V, VI, and VIII.

Afterwards, a number of other 15th-century German writings are presented or mentioned briefly:

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[11] These are only described in words by Folkerts, but see the depictions in Curtze 1895, 232ff., 278–280; Cajori 1928–1929, 1.95ff.; and Tropfke, Vogel, *et alii* 1980, 281.

[12] Thus not only Folkerts but also the re-drawings in Tropfke, Vogel, *et alii* 1980, 206 and Vogel 1954, Tafel VI.

[13] This shape is found, e.g., in Vatican Library, Chigiana, M.VIII.170, written in Venice in *ca* 1395. A reduction of the equally classical shape 𝔪̃ is definitely less likely.

[14] Reproduced in high resolution on the webpage: http://columbia.edu/cgi-bin/dlo?obj=ds.Columbia-NY.NNC-RBML.6662&size=large].

○ the (mostly non-algebraic) problems added to the *Algorismus ratis-bonensis* by Fridericus Amann and the algebra written by Amann in 1461 (both Bayerische Staatsbibliothek, Clm 14908);[15]

○ from Dresden, C 80, a 'Latin algebra' as well as a 'German algebra' from 1481 which 'seems to depend on the "Latin algebra"' [XII.9];[16]

○ marginal notes in the same manuscript made by Johannes Widmann, and the same author's *Behende und hubsche Rechenung auff allen kauffmanschafft* from 1489;

○ the writings of Andreas Alexander (b. *ca* 1470), a pupil of a certain Aquinas (an otherwise obscure Dominican friar from whom Regiomontanus says that he has learned);

○ the *Initius algebras* which *may* have been written by Alexander or by Adam Ries;

○ Ries' (non-algebraic) *Rechenbuch* as well as the two editions of his *Coss* [1524, 1543+];

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[15] The problems were published in Vogel 1954; the algebra, in Curtze 1895, 49–73.

[16] The former was published in Wappler 1887; the latter, in Vogel 1981. The codex was in the possession of Widman, and the *Latin algebra* was used by him. Since the *German algebra* makes abundant use both of a fraction-like notation for monomials known from Italian writings [see below, text around 137n 21] and of the phrases '*mach mir die rechnung*'/'*Und moch des gleichen rechnung alzo*' corresponding to the Italian '*fammi questa ragione*'/'*così fa le simiglianti*', none of which are found in the *Latin algebra*, the *German algebra* must either draw on several sources of inspiration, or it must share a precursor with the *Latin algebra* rather than depend on it (or both).

That it must depend on several sources was indeed already observed by Vogel [1981, 10]. To Vogel's observations can now be added not only that the fraction-like notation for monomials is of Italian origin but also that the strange term and abbreviation for the fourth power (*wurczell von der worczell*/'root of the root') looks like a crossbreed between Piero's negative powers and Pacioli's alternative notation [see 133n10, above]. The idea to provide the fifth case (the one with a double solution) with three examples also corresponds to what can be found in Italy (Jacopo da Firenze as well as Dardi)—the original point being that one case requires the additive solution, one the subtractive solution, and one is satisfied by both.

The use of 'root of root' in passages of the German algebra that are parallel to passages where the *Latin algebra* has the regular repeated *zensus*-abbreviation ʓʓ suggests that these parallels are due to the sharing of a common source rather than to direct translation.

○ Rudolff's *Coss* [1525] and Stifel's *Arithmetica integra* [1544]; and
○ the Cistercian Conrad Landvogt (*ca* 1450 to 1500+), whom Folkerts himself has brought to light.

Folkerts bases his claim regarding Regiomontanus' central role in the transmission on various pieces of evidence. First, the algebraic problems in the Plimpton collection have the heading *Regule de cosa et censo sex sunt capitula, per que omnis computatio solet calculari*; whereas Amann gives the title *Regule dela cose secundum 6 capitola*. The similarity is not striking. Moreover, if Amann *had* copied Regiomontanus, he would have had no reason whatever to restore Italian grammar (*dela cose* instead of *de cosa*). A close common source, however, is very likely.[17]

Second, Regiomontanus is supposed to have invented his own symbolism; and Amann, to have borrowed it. For, given that Amann appears to have visited Vienna in 1456, Folkerts thinks that 'there are good reasons to assume that he met Regiomontanus there and at this meeting... learnt of his symbols' [XII.8]. (Regiomontanus was 20 years old by then, while Amann must have been close to 50). Amann's symbols for *res/cosa* and *zensus* are indeed fairly similar to those of Regiomontanus. However, in V.201ff., Folkerts indicates that parts of the Plimpton manuscript which appear *not* to be written by Regiomontanus also use symbols and that one section uses exactly the same symbols as Regiomontanus. There Folkerts points out that this might represent a precursor to Regiomontanus' symbolism. In that part of the Plimpton text, it is true, the symbols are not superscript, but even this is hardly an innovation due to Regiomontanus (nor is it, as we have observed, a constant habit of his): superscript symbols following the coefficient (the square meaning *censo* sometimes above, but *co* for *cosa* always following) were

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[17] Indeed, the two examples from Regiomontanus' text which are reproduced on the web [see 134n14, above] coincide substantially with those of Amann— much more so, indeed, than they would have done if Amann had reproduced from memory what he had discussed with Regiomontanus (see imminently), but much less than if he had translated from the Plimpton manuscript. One difference is informative. In Regiomontanus' text, there is a reference to the principle that when equals are added to equals, equals result. This Euclidean argument for the traditional *restoration* operation is absent from Amann's text, and thus likely to be Regiomontanus' own contribution—and an early manifestation of his characteristic approach.

also used by Pacioli in a manuscript finished in 1478 (Vatican, Vat.
Lat. 3129), which also (for example, on fol. 67v) uses the horizontal
stroke as an equation sign (but ⓜ for minus).[18] Since superscript □
and *co* (and sometimes *cen* for *censo*) written above the coefficient
are also used in the Italian manuscript Vat. Lat. 10488 of 1424, for
instance, on foll. 36v, 38v, 92r–v (original foliation), it is clear that
Pacioli did not take his inspiration from Regiomontanus.[19] Ultimately,
this notation is likely to be a borrowing from Maghreb algebra.[20]

   A different, fraction-like notation was used by Dardi of Pisa,[21]
and also in the draft manuscript *Trattato di tutta l'arte dell'abacho*
from *ca* 1334: $\frac{12}{c}$ stands for 12 *cose*, $\frac{4}{c}$ for 4 *censi*. The same notation
is used in the *German algebra* in C 80.[22] All in all, it is possible though
not certain that some later cossists learned their symbolism (or part
of it) from Regiomontanus. It is certain, however, that not all of
them did, and equally certain that Regiomontanus did not invent it.

   Third, it is said on XII.9 that the

   order of the [equation] types, which is elsewhere varied, is
   the same in the 'German algebra' in MS C 80 and in the
   Regiomontanus text in MS Plimpton 188. This cannot be a
   coincidence.

Evaluation of this statement is difficult since Folkerts gives no exact
information about the presentation of the cases in MS Plimpton 188.
However, in VIII.418, it is stated that

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[18] For a discussion of the stroke as equation sign in Pacioli's *Summa*, see Cajori
1928–1929, 1.110ff.

[19] Vat. Lat. 10488 sometimes uses ⓜ, sometimes ⓜₑ for minus.

[20] Cf., e.g., Tropfke, Vogel, *et alii* 1980, 376.

[21] Høyrup [2007a, 170] argues that this symbolism, found in the two earliest
manuscripts, was already used in Dardi's original from 1344.

[22] With a set of symbols for the algebraic powers which is neither identical with
what can be found in Italian treatises nor with those of Regiomontanus,
Amann, or the *Latin algebra*; see the facsimiles in Vogel 1981, Tafel 1–3,
and the comparison in Vogel 1981, 11 (where it should be observed that the
symbolic notation ascribed to Robert of Chester and the year 1150 refers to
marginal notes in C 80 and to an appendix to Robert's translation found in
15th-century manuscripts from the South-German area).

   In the very last problem of the *German algebra* [Vogel 1981, 43], a
different (but equally Italian) notation is used: a superscript *c* (for *cosa*),
above or following the coefficient.

the Latin text in the Plimpton manuscript, which describes
the six forms of equations, agrees word-for-word with the
German translation that Fridericus Amann wrote five years
later.

But this simply means that the order for these six fundamental cases
is the standard order of Italian *abbaco* algebra—which certainly dif-
fers from the order of al-Khwārizmī, Abū Kāmil, and Fibonacci [see
Curtze 1895, 50]. The same order is found in the *Latin algebra* as
well as in the *German algebra* from C 80. Such agreement concern-
ing the fundamental cases thus only indicates common roots in the
abbaco tradition and nothing more.

Then, there are 18 more cases, which are either homogeneous or
reducible to the second degree. These cases are found in the *Latin
algebra* [Wappler 1887, 12ff.] as well as in the *German algebra* [Vogel
1981, 22] from C 80.[23] These cases share not only their order (which is
unusual and may perhaps be of Italian origin) but also the numerical
parameters. *This* is certainly not be a coincidence, even though the
cases themselves were all familiar in *abbaco* algebra since the early
14th century. Regiomontanus also has 18 more cases, and most of
them coincide with those of the two algebras from C 80 and follow
the same order. But, if Folkerts' transcription in modern symbols in
V.n150 is reliable, two cases are different:

○ no. 12 is $ax^4 + bx^2 = cx^3 + dx^2$, while agreement with the algebras
  in C 80 would require $ax^4 = cx^3 + dx^2$;
○ no. 14 is $ax^2 = \sqrt{b}$, whereas agreement would demand $ax^2 = \sqrt{bx^2}$.

The latter deviation *might* be a miswriting due to Folkerts or his
typographer, but the former is not. So, once more, the evidence
suggests shared inspiration rather than copying from Regiomontanus.

Summing up, Folkerts' description of 15th-century German alge-
bra is certainly indispensable for any further discussion of the topic
in that it lists all known important and several (though not all) mi-
nor manuscript sources and points to many of the parameters that
have to be taken into account. Thus, it was only through the use
of Folkerts' text that I was able to grasp and sift the material well

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[23] The *Latin algebra* has one more case, which is corrupt and lacks an illus-
trating example), and which its compiler claims he 'found elsewhere' (*alibi
inveni*) [Wappler 1887, 12].

enough to formulate my objections. In my view, Folkerts' conclusion is premature and sometimes contradicted by precise inspection of the sources. In consequence, I believe it to be mistaken: Italian *abbaco* algebra appears to have inspired and spurred the German development not through a single but through multiple channels.[24] However, no definite conclusions should be drawn before manuscripts are gauged against the essential parameters both on the Italian and the German side. Unfortunately, few of the printed editions of Italian *abbaco* manuscripts that have been published during the last 50 years have bothered much about symbolism-like abbreviations and non-geometric marginal diagrams. It is to be hoped, then, that Folkerts' overview may contribute to changing this state of affairs!

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[24] Further evidence for this, beyond the parameters discussed by Folkerts and the observations made in 135n16 above, comes from the German standard spellings *coss*, *zensus* and *unze*. They point to inspiration from northern Italy [cf. Rohlfs 1966–1969, I.201f., 284, 388], where *cossa/chossa*, *zenso* and *onzia* are common, say, to Genoa rather than to Venice. Regiomontanus, in MS Plimpton 188, writes *cosa*; the *German algebra* has *cossa*. The *Latin algebra*, as mentioned in 135n16 above, uses an abbreviation for the second power (ʒ) which is derived from '*zensus*'.

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